

A theoretical analysis of the resolution due to diffusion and size-dispersion of particles in deterministic lateral displacement devices

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We present a model including diffusion and particle-size dispersion for separation of particles in deterministic lateral displacement devices also known as bumper arrays. We determine the upper critical diameter for diffusion-dominated motion and the lower critical diameter for pure convection-induced displacement. Our model explains the systematic deviation, observed for small particles in several experiments, from the critical diameter for separation given by simple laminar flow considerations.

I. INTRODUCTION

In 2004 Huang et al. [1] developed the elegant method of particle separation by deterministic lateral displacement in so-called microfluidic bumper arrays. The method, which relies on the laminar flow properties characteristic of microfluidics, shows a great potential for fast and accurate separation of particles on the micrometer scale [1, 2, 3, 4]. Among the key assets of the deterministic lateral displacement separation principle are that clogging can be avoided because particles much smaller than the feature size of the devices can be separated, that the devices are passive, i.e. the particles bump into solid obstacles or bumpers, and that the separation process is continuous.

More precisely, particle transport in microfluidic bumper arrays is primarily governed by convection due to the fluid flow and displacement due to interaction with the bumpers in the array [1]. These processes are deterministic and the critical diameter for separation of relatively large particles in these devices is well understood in terms of the width of flow lanes bifurcating around the bumpers in the periodic arrays [3]. However, if bumper arrays and particles are scaled down, diffusion will influence the separation process and affect the critical particle size significantly. Previously reported data on separation of particles in bumper arrays all show a bias towards larger critical particle size than that given by the width of the flow lanes nearest to the bumpers of the array [1, 2, 3]. In this work we extend existing models by adding diffusion and taking particle-diameter dispersion into account, and thereby explain the observed discrepancy.

In bumper arrays particles are convected by the fluid flow through an array of bumpers placed in columns separated by the distance λ in the flow direction, see Fig. 1(a). For a given integer N , The array is made N -periodic in the flow direction by displacing the bumpers in a given column a distance λ/N perpendicular to the flow direc-

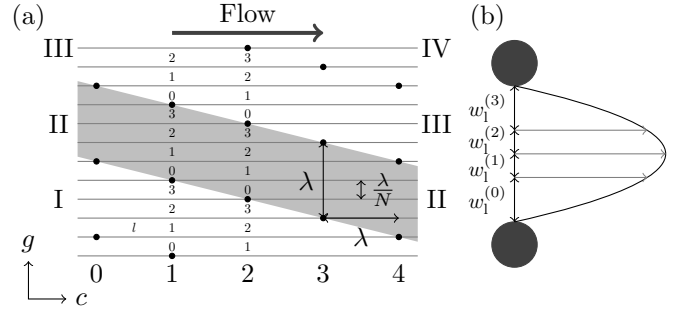


FIG. 1: (a) An array of bumpers (black dots) with the definition of the lane number l (small arabic numbers), the column number c (large arabic numbers), and the gap number g (roman numbers). The shaded region illustrates how the shift in the position of gap II follows the geometry of the array. (b) Close-up of a single gap between two bumpers (disks) in the array. Each of the four flow lanes carries the same flow rate. Due to the parabolic flow profile in the gap region, the width $w_l^{(l)}$ of flow-lane l depends on the position in the gap.

tion with respect to the bumper positions in the previous column. Due to this periodicity of the array and the laminarity of the flow, the stream can naturally be divided into N lanes, each carrying the same amount of fluid flux, and each having a specific path through the device, see Ref. [1].

For a given steady pressure drop, the fluid in the device moves with an average velocity u_0 . Assuming a parabolic velocity profile $u(x)$ in the gap of width w_g between two neighboring bumpers, see Fig. 1(b),

$$u(x) = 6u_0 \frac{x}{w_g} \left(1 - \frac{x}{w_g}\right), \quad (1)$$

the total flow rate Q_{tot} is given by

$$Q_{\text{tot}} = \int_0^{w_g} u(x) dx = w_g u_0. \quad (2)$$

For an N -periodic array, the N flow lanes in a given gap carry the same flow rate Q_{tot}/N . The width $w_l^{(l)}$ of lane l

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is found by solving

$$\frac{Q_{\text{tot}}}{N} = \int_{x^{(l)}}^{x^{(l)} + w_1^{(l)}} u(x) dx, \quad (3)$$

where $x^{(l)} = \sum_{j=0}^l w_1^{(j)}$ is the starting position of lane l . In the simple bifurcating flow-lane model [1, 3] the critical diameter d_c is given as $d_c/2 = w_1^{(1)}$. A small particle with $d < d_c$ will never leave its initial flow lane and will thus be convected in the general flow direction following a so-called zigzag path. Large particles with $d > d_c$ will quickly bump against a bumper and from then on be forced by consecutive bumping to follow the skew direction of the array geometry, the so-called displacement path. When a particle gets bumped by a bumper in the array it will be displaced perpendicular to the flow direction until its center is located one particle radius $d/2$ from the surface of the bumper. This corresponds to n_1 lanes of displacement,

$$n_1 = \frac{N}{w_g u_0} \int_0^{d/2} u(x) dx = N \frac{d^2}{4w_g^2} \left(3 - \frac{d}{w} \right). \quad (4)$$

In the bulk fluid, where the lanes are assumed to have equal width λ/N , see Fig. 1(a), the displaced distance ℓ_{disp} is therefore

$$\ell_{\text{disp}}(d) = n_1 \frac{\lambda}{N} = \lambda \frac{d^2}{4w_g^2} \left(3 - \frac{d}{w_g} \right). \quad (5)$$

In this work we extend the simple bifurcating flow-lane model by including diffusion and particle-diameter dispersion.

II. MODEL INCLUDING DIFFUSION

During the average time $\tau = \lambda/u_0$ it takes a particle to move by convection from one column to the next, it also diffuses. We assume that the diffusion process perpendicular to the flow direction is normally distributed with mean value zero and variance

$$\sigma^2 = 2D\tau, \quad (6)$$

where the diffusivity D is given by the Stokes–Einstein expression

$$D = \frac{k_B T}{3\pi\eta d}. \quad (7)$$

Here k_B is Boltzmann's constant, T is the temperature and η is the viscosity of the fluid.

A. Diffusion model

In order to escape bumping, a particle must in the time interval τ diffuse more than the difference $\ell_{\text{disp}} - \lambda/N$

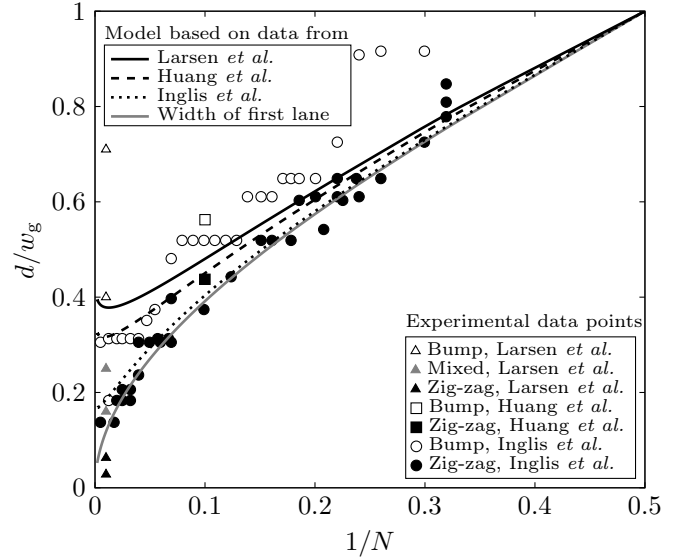


FIG. 2: Our model applied to the experimental data of Inglis et al. [3], Huang et al. [1] and Larsen [5]. Particle diameter d over gap width w_g is plotted as a function of the inverse period $1/N$. The black lines show the theoretically predicted critical particle size for the bumper arrays used by Inglis et al. [3], Huang et al. [1] and Larsen [5], respectively. The experimental data points are representing particles following bumper paths (open symbols), zigzag paths (solid black symbols), and neither of these paths (solid gray symbols).

between the bulk displacement and the shift in position of the next bumper. The probability p_{esc} for this to happen is given in terms of the error function as

$$p_{\text{esc}}(d) = \frac{1}{2} - \frac{1}{2} \operatorname{erf} \left(\frac{\ell_{\text{disp}}(d) - \frac{1}{N} \lambda}{\sqrt{2} \sigma(d)} \right), \quad (8)$$

where we have introduced the d -dependence explicitly. When a particle is transported through a bumper array it must bump at every bumper within one period of the array in order to be displaced one gap at the outlet. Thus, if the particle escapes at least one time in N attempts, it will not be displaced. We define the critical particle size d_c to be the size for which half of the particles escape bumping as they pass one period of the array. Thus d_c can be found by solving

$$\sum_{k=1}^N \binom{N}{k} p_{\text{esc}}(d_c)^k \left[1 - p_{\text{esc}}(d_c) \right]^{N-k} = \frac{1}{2}. \quad (9)$$

In Fig. 2 we have plotted the critical particle size as a function of the bumper period for parameter values corresponding to the bumper arrays used by Huang et al. [1] (dashed line), Inglis et al. [3] (dotted line), and Larsen [5] (full line).

B. Comparison to experiments

The observation that the critical particle size in a bumper device is larger than the width of the first flow lane is also supported by the experimental data in Fig. 2 of Ref. [3]. Our model suggests that the deviation of the critical particle size from the width of the first flow lane can be explained by diffusion of the particles. In Fig. 2 above, it is seen how well the lines are predicting the transition between zigzag paths and displacement paths: the thick line is dividing open and closed triangles, the dashed line is dividing open and closed squares, and, to a lesser extent, the dotted line is dividing open and closed circles.

Using parameter values corresponding to the bumper device presented by Huang et al. [1] our model predicts a critical particle diameter of 0.45 times the width of the gap for the particles traveling through the device with an average velocity of $400 \mu\text{m/s}$. This is in good agreement with Fig. 2(a) in Ref. [1].

III. A DISCRETE MODEL INCLUDING DIFFUSION AND DISPERSION

The particles used in the experiments on separation of particles are not mono-disperse. Their average diameters are distributed around a certain mean value with a relative standard deviation $\Delta d/d$, which typically is 20 %, 10 % and 5 % for particles with $d = 25 \text{ nm}$, $d = 100 \text{ nm}$ and $d = 500 \text{ nm}$, respectively.

In the following we introduce a discrete model of the transport of particles with different diameters d in an N -periodic bumper array taking convection, diffusion and size-dispersion into account. The model allows us to study the relative influence of all three phenomena on the separation efficiency in a fast and simple manner. In particular our results suggest that the critical size for separation or displacement, studied above, must be supplemented by a smaller critical size below which pure diffusion governs the motion of the particles in the bumper array. This prediction has not yet been tested experimentally.

A. Definition of the discrete model

At any instant, a particle is assumed to be positioned in the center of a flow lane l situated in a gap g in some column c of the array. For simplicity we further assume that the size distribution of any given set of particles is a normal distribution with a mean value given by the size quoted by the manufacturer and a relative standard deviation of 10 %.

By convection any given particle moves from one column to the next. If it ends up in the last lane ($l = N - 1$) in one gap, it will be shifted to the first lane ($l = 0$) in the subsequent gap. Otherwise it will stay in the current

gap and move up one lane. In our model pure convection is therefore described by the discrete map

$$(c, g, l) \mapsto \begin{cases} (c + 1, g + 1, 0) & , \text{ if } l = N - 1, \\ (c + 1, g, l + 1) & , \text{ otherwise.} \end{cases} \quad (10)$$

Because of the finite diameter d of the particle there is a minimum and a maximum lane number that it can occupy. The minimum lane number is the smallest integer l_{\min} that satisfies

$$\sum_{l=0}^{l_{\min}} w_l^{(l)} > \frac{d}{2}. \quad (11a)$$

Similarly, the maximum lane number l_{\max} is the largest integer that satisfies

$$\sum_{l=l_{\max}}^{N-1} w_l^{(l)} > \frac{d}{2}. \quad (11b)$$

Consequently, the simple convection mapping from Eq. (10) needs to be modified to account for the finite size of the particles

$$(c, g, l) \mapsto \begin{cases} (c + 1, g + 1, l_{\min}) & , \text{ if } l = N - 1, \\ (c + 1, g, l + 1) & , \text{ if } l < l_{\max} - 1, \\ (c + 1, g, l_{\max}) & , \text{ otherwise.} \end{cases} \quad (12)$$

The above convection scheme accounts for the separation of particles in deterministic lateral displacement devices according to size. The critical particle radius predicted by this model is

$$d_c = 2w_1^{(0)} \quad (13)$$

in accordance with the geometric arguments of Ref. [3].

To characterize the separation quantitatively, we define the relative particle numbers r_0 , r_1 and r_d as

$$r_0 = \text{relative number of particles following the zigzag path.} \quad (14a)$$

$$r_1 = \text{relative number of particles following the displacement path.} \quad (14b)$$

$$r_d = \text{relative number of all other particles.} \quad (14c)$$

With these definitions the sum $r_0 + r_1 + r_d$ is always unity. If $r_0 = 1$ all particles follow the zigzag path and if $r_1 = 1$ all particles follow the displacement path. If $r_d \neq 0$ some of the particles end up at positions not explained by the deterministic analysis of the separation process. In Fig. 3 we have plotted the relative particle numbers r_0 , r_1 , and r_d as a function of average particle diameter d .

B. Pure mono-disperse convection

For mono-disperse and non-diffusing particles, the model results, as expected, in two modes: the zigzag

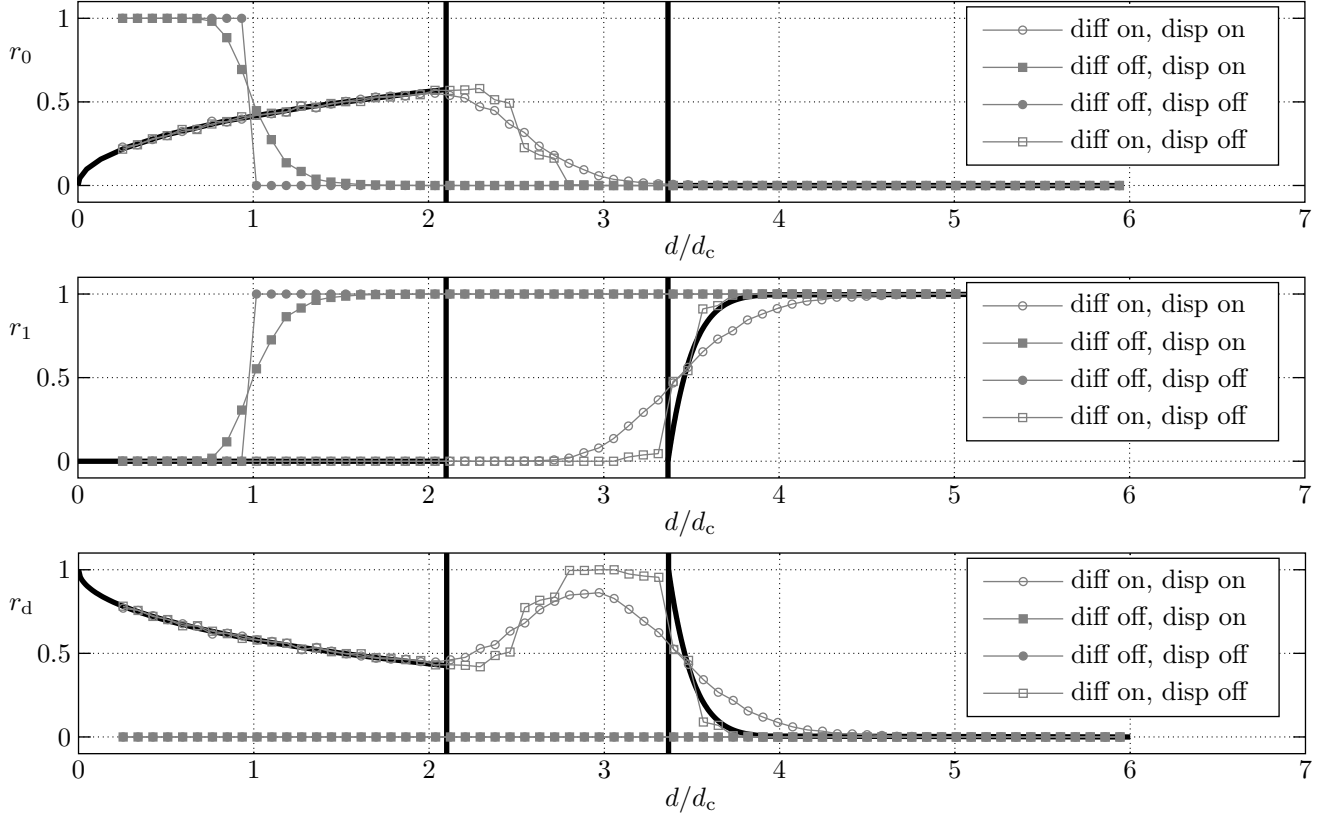


FIG. 3: Relative number r_0 , r_1 and r_d of particles following the zigzag path, the displacement path, and neither of these two paths, respectively, plotted as a function of the normalized, average particle diameter d/d_c . The parameters of the bumper array is taken from Ref. [5]: $N = 100$, $\lambda = 8 \mu\text{m}$, $w_g = 1 \mu\text{m}$, $L = 20N\lambda = 16 \text{ mm}$, and $u_0 = 250 \mu\text{m/s}$. The buffer liquid is water at room temperature. Neglecting diffusion (solid symbols), the particles follow the zigzag path if $d < d_c = 118 \text{ nm}$. If $d > d_c$ they follow the displacement path. Including diffusion (open symbols), the small particles are not influenced by the bumpers. For $d \gtrsim 2.1 d_c$ the influence of the bumpers set in, and for $d \gtrsim 3.4 d_c$ the particles follow the displacement path. The thick black vertical lines indicate the particle size when small particles stop behaving purely diffusive (left-most lines) and when large particles begin a purely deterministic displacement (right-most lines).

mode and the displacement mode, see the closed circles in Fig. 3. For $d < d_c$ we have $r_0 = 1$, and for $d \geq d_c$ we have $r_1 = 1$, while we always have $r_d = 0$. The relative particle numbers can therefore be written as

$$(r_0, r_d, r_1) = \begin{cases} (1, 0, 0) & \text{for } d < d_c \\ (0, 0, 1) & \text{for } d \geq d_c \end{cases}. \quad (15)$$

C. Influence of size dispersion

If we assume that the particles are not mono-disperse, but are distributed around a mean size d with standard deviation Δd , the shift as a function of d from the zigzag mode to the displacement mode happens gradually instead of abruptly at a certain critical size d_c (Fig. 3, closed squares). The relative number of particles following the zigzag path r_0 can be found by integrating over all particle sizes smaller than the critical diameter given

by the array geometry

$$r_0 = \int_{-\infty}^{d_c} \frac{1}{\sqrt{2\pi}(\Delta d)^2} \exp\left(\frac{-(s-d)^2}{2(\Delta d)^2}\right) ds. \quad (16a)$$

Similarly, the relative number of particles following the displacement path r_1 can be found by integrating over all particle sizes larger than d_c

$$r_1 = \int_{d_c}^{\infty} \frac{1}{\sqrt{2\pi}(\Delta d)^2} \exp\left(\frac{-(s-d)^2}{2(\Delta d)^2}\right) ds. \quad (16b)$$

The system is still a bi-modal system because $r_d = 0$ for all particle sizes.

D. Influence of diffusion

In 1D during the time step τ a particle diffuses the distance ℓ , the average of which is the size-dependent

diffusion length $\sigma(d)$ given by

$$\sigma(d) = \langle \ell \rangle = \sqrt{2D\tau} = \sqrt{2 \frac{k_B T}{3\pi\eta d} \frac{\lambda}{u_0}}. \quad (17)$$

In our model we discretize the transverse diffusion as the properly rounded number of flow lanes crossed by the particle during diffusion,

$$n_{\text{jump}} = \frac{N}{\lambda} \ell. \quad (18)$$

The addition of diffusion smears out the displacement of the particles and causes the critical radius to be larger than in the diffusion-less case (Fig. 3, open symbols).

1. Bumping criterion for small particles

Very small particles are completely dominated by diffusion, and the particle distribution at the end of the array is simply given by the transverse diffusion of the particles during the time L/u_0 it takes for the particle to be convected all the way L through the array. For small particles we therefore have

$$r_0 = \int_{-\frac{3}{2}\frac{\lambda}{N}}^{\frac{3}{2}\frac{\lambda}{N}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx, \quad (19)$$

where $\sigma^2 = 2DL/u_0$.

As the particle diameter d is increased, the bumpers begin to become important as the diffusion length $\sigma(d)$, Eq. (17), is decreased and becomes equal to the displacement length ℓ_{disp} , Eq. (5). Using the criterion $\sigma(d_1) = \ell_{\text{disp}}$, with the parameter values used in Fig. 3, we find that particles stop behaving as small diffusion-dominated particles and start interacting with the bumpers when $d_1 = 2.1 d_c$. This cross-over value is indicated by the left-most vertical lines in Fig. 3, and it fits well with the simulation data.

2. Bumping criterion for large particles

Large particles will interact with the bumpers at every row in the array and their position is thus reset at every bump to be ℓ_{disp} . Diffusion can therefore be neglected for such particles, they all follow the displacement path, and $r_1 = 1$.

As the particle diameter is lowered, the probability p_{esc} that a particle escapes the displacement path can be estimated by the probability of diffusing from the displaced position to the last flow lane in the gap, i.e. the distance $\ell_{\text{disp}} - \frac{\lambda}{N}$. This probability p_{esc} is given by Eq. (8).

In order to follow the displacement path, a particle must bump at each row in the array. If we consider a N -periodic array with m full periods, the particles will have

mN bumping opportunities as they pass through the entire array. If a particle evades bumping at a bumper, it will be convected by the flow through a full period of the array before bumping is possible again. Because of the escape, it will miss N bumping opportunities and end up one gap from the displacement path. Consequently, if a particle escapes one time, it will only have $(m-1)N$ bumping opportunities and has therefore escaped bumping with a probability of $1/[(m-1)N]$. The upper critical particle size d_2 for convection-induced displacement is defined using Eq. (8) as

$$p_{\text{esc}}(d_2) = \frac{1}{(m-1)N}. \quad (20)$$

For the device used in the experiments by Larsen [5] we find $d_2 = 3.4 d_c$. The predicted upper limit for diffusion dominated motion and the lower limit for convection-induced displacement compares well with the experimental observation by Larsen [5] that some particles end up between the displacement path and the zigzag path (Fig. 2, gray triangles).

IV. CONCLUSION

Experimental data on separation of particles in bumper arrays all show a systematic deviation from predictions made from the bifurcating flow lane model [1, 3, 5]. By adding diffusion to the bifurcating flow lane model, we explain the discrepancy. In addition, we have suggested a simple discrete model for simulating particle separation in bumper arrays. The presented model takes particle diffusion and size dispersion into account and has been validated against experimental data for a bumper device with period $N = 100$. The transition from zigzag path to displacement path happens at a particle size 2.1 to 3.4 times larger than the critical particle size predicted from geometrical arguments. This is in correspondence with the experimental observations from Larsen [5]. Our discrete model and the estimates presented in this paper suggest that particles of twice the size of the geometrical critical size of the $N = 100$ bumper device behave diffusively and are not affected by the bumpers because the small diffusive particles rarely come into contact with the bumpers due to random Brownian motion.

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